

Dynamics for a simple graph using the $U(N)$ framework for loop quantum gravity

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Abstract. The implementation of the dynamics in loop quantum gravity (LQG) is still an open problem. Here, we discuss a tentative dynamics for the simplest class of graphs in LQG: Two vertices linked with an arbitrary number of edges. We find an interesting global $U(N)$ symmetry in this model that selects the homogeneous/isotropic sector. Then, we propose a quantum Hamiltonian operator for this reduced sector. Finally, we introduce the spinor representation for LQG in order to propose a classical effective dynamics for this model.

1. Introduction

Loop quantum gravity (LQG) proposes a non-perturbative mathematical formulation of the kinematics of quantum gravity. The Hilbert space is generated by states defined over oriented graphs whose edges are labeled by irreducible representations of the $SU(2)$ group and whose vertices are decorated with intertwiners ($SU(2)$ invariant tensors). These are the so-called spin networks. Despite the several advances that have taken place in this field, one of the main challenges faced by the theory is the systematic implementation of the dynamics. Our goal is to focus on a specific model in order to propose a suitable Hamiltonian for it.

We use the $U(N)$ framework for $SU(2)$ -intertwiners [1, 2, 3] to study the spin network Hilbert space of the 2-vertex graph (2 nodes joined by an arbitrary number N of links) from a new point of view. We identify a global symmetry that selects a homogeneous and isotropic sector of this system [4] and we construct the operators that leave this sector invariant. They will be the building blocks to construct the Hamiltonian operator.

On the other hand, the recent spinor representation for LQG [5, 6, 7] opens a new way to study several aspects of LQG. We apply this new formalism to the 2-vertex graph and we propose a classical action with an interaction term which encodes the effective dynamics of this

system. This interaction term is, indeed, the classical counterpart of the quantum Hamiltonian obtained within the $U(N)$ framework.

2. The $U(N)$ framework

The $U(N)$ framework introduced in [1, 2] is very useful to study the Hilbert space of intertwiners with N legs and to build appropriate semi-classical states [3]. The basic tool is the Schwinger representation of the $\mathfrak{su}(2)$ Lie algebra in terms of a pair of harmonic oscillators a and b :

$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b), \quad J_+ = a^\dagger b, \quad J_- = ab^\dagger.$$

Labeling the N legs with the index i , we identify $SU(2)$ invariant operators acting on pairs of (possibly equal) legs i, j [1, 3]:

$$E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j, \quad (E_{ij}^\dagger = E_{ji}), \quad F_{ij} = a_i b_j - a_j b_i, \quad (F_{ji} = -F_{ij}).$$

The operators E form a $\mathfrak{u}(N)$ -algebra and they also form a closed algebra together with the operators F, F^\dagger . Notice that the diagonal operators give the energy on each leg, $E_{ii} = E_i$, which gives twice the spin j_i of the $\mathfrak{su}(2)$ representation carried by that leg. This spin j_i is identified geometrically as the area associated to the leg i and the total energy $E = \sum_i E_i$ gives twice the total area $J = \sum_i j_i$ associated to the intertwiner. The E_{ij} -operators change the energy/area carried by each leg, while still conserving the total energy, while the operators F_{ij} (resp. F_{ij}^\dagger) will decrease (resp. increase) the total area E by 2:

$$[E, E_{ij}] = 0, \quad [E, F_{ij}] = -2F_{ij}, \quad [E, F_{ij}^\dagger] = +2F_{ij}^\dagger.$$

The operators E_{ij} allow then to navigate from state to state within each subspace $\mathcal{H}_N^{(J)}$ of N -valent intertwiners with fixed total area J ; and the operators F_{ij}^\dagger and F_{ij} allow to go from one subspace $\mathcal{H}_N^{(J)}$ to the next $\mathcal{H}_N^{(J\pm 1)}$, thus endowing the full space of N -valent intertwiners with a Fock space structure with creation operators F_{ij}^\dagger and annihilation operators F_{ij} . Besides, it was proven [2] that each subspace $\mathcal{H}_N^{(J)}$ carries an irreducible representation of $U(N)$ generated by the E_{ij} operators. Finally, it is worth pointing out that the operators $E_{ij}, F_{ij}, F_{ij}^\dagger$ satisfy certain quadratic constraints, which correspond to a matrix algebra [4].

3. The 2 vertex model and the quantum Hamiltonian

We consider the simplest class of non-trivial graphs for spin network states in LQG: a graph with two vertices linked by N edges, as shown in fig.1.

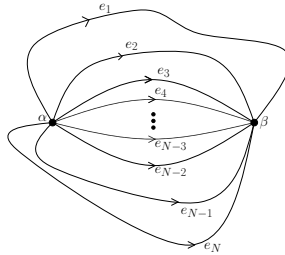


Figure 1. The 2-vertex graph with vertices α and β and the N edges linking them.

There are matching conditions [2] imposing that each edge carries a unique $SU(2)$ representation (same spin seen from α and from β). This translates into an equal energy condition:

$$\mathcal{E}_i \equiv E_i^{(\alpha)} - E_i^{(\beta)} = 0.$$

These constraints \mathcal{E}_k turn out to be part of a larger $U(N)$ symmetry algebra. Indeed, we introduce the more general operators:

$$\mathcal{E}_{ij} \equiv E_{ij}^{(\alpha)} - E_{ji}^{(\beta)} = E_{ij}^{(\alpha)} - (E_{ij}^{(\beta)})^\dagger,$$

that form a $U(N)$ algebra and that reduce to the matching conditions in the case $i = j$. Now, one can show [4] that by imposing the global $U(N)$ -invariance generated by the \mathcal{E}_{ij} 's on our 2-vertex system, one obtains a single state $|J\rangle$ for each total boundary area J . Thus, the $U(N)$ invariance is restricting our system to states that are homogeneous and isotropic (the quantum state is the same at every point of space, i.e. at α and β , and all directions or edges are equivalent).

We propose a dynamics for this system compatible with the $U(N)$ -invariance. Investigating the structure of the $U(N)$ -invariant operators, we propose the most general $U(N)$ invariant Hamiltonian (allowing only elementary changes in the total area), up to a renormalization by a E -dependent factor:

$$H \equiv \lambda \sum_{ij} E_{ij}^{(\alpha)} E_{ij}^{(\beta)} + \left(\sigma \sum_{ij} F_{ij}^{(\alpha)} F_{ij}^{(\beta)} + \bar{\sigma} \sum_{ij} F_{ij}^{\alpha\dagger} F_{ij}^{\beta\dagger} \right), \quad (1)$$

where the coupling λ is real while σ can be complex a priori, so that the operator H is Hermitian.

We studied the properties of this Hamiltonian on the $U(N)$ invariant Hilbert space. Its action on states $|J\rangle$ is known and its spectral properties have been analyzed [4]. It turns out that it shares several mathematical analogies with the evolution operator used in loop quantum cosmology. At the physical level, interpreted as a cosmological model, this simple dynamical 2-vertex model also leads generically to a big bounce and avoids the big bang singularity.

4. Spinors and effective dynamics

Based on the Schwinger representation of $SU(2)$ in terms of harmonic oscillators, it is possible to give a representation of the classical phase of LQG in terms of spinor variables [5, 7]. The quantization of this classical system will lead us back to the $U(N)$ framework for intertwiners. Focusing on this classical system, we write an action principle with an effective dynamics of the spinors reflecting the quantum dynamics defined above.

Let us start by introducing the usual spinor notation. Let us define the spinor z and its dual:

$$|z\rangle = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \in \mathbb{C}^2, \quad \langle z| = \begin{pmatrix} \bar{z}^0 & \bar{z}^1 \end{pmatrix}, \quad |z] \equiv \begin{pmatrix} -\bar{z}^1 \\ \bar{z}^0 \end{pmatrix}.$$

In order now to describe N -valent intertwiners, we consider N spinors z_i satisfying a closure condition¹ that, in terms of their components, is given by:

$$\sum_i |z_i\rangle \langle z_i| \propto \mathbb{I} \Leftrightarrow \sum_i z_i^0 \bar{z}_i^1 = 0, \quad \sum_i |z_i^0|^2 = \sum_i |z_i^1|^2 = \frac{1}{2} \sum_i \langle z_i | z_i \rangle. \quad (2)$$

Solutions are parameterized in terms of a positive number $\lambda \in \mathbb{R}_+$ and a unitary matrix $u \in U(N)$ up to $U(N-2) \times SU(2)$ right-transformations with $z_i^0 = \sqrt{\lambda} u_{i1}$ and $z_i^1 = \sqrt{\lambda} u_{i2}$.

¹ We associate to each spinor z_i a 3-vector $\vec{X}(z_i) = \langle z_i | \vec{\sigma} | z_i \rangle$ by projecting it onto the Pauli matrices. Then the closure constraint is $\sum_i \vec{X}(z_i) = 0$ and we identify $\vec{X}(z_i)$ as the normal vector to the dual surface to the leg i .

The phase space is defined by the canonical Poisson bracket $\{z_i^a, \bar{z}_j^b\} \equiv i\delta^{ab}\delta_{ij}$. The quantization will be promoting z_i and \bar{z}_i as the annihilation and creation operators of harmonic oscillators. Then the classical matrices $M_{ij} = \langle z_i | z_j \rangle$ and $Q_{ij} = [z_j | z_i]$ are the classical counterparts of the operators E and F .

The $U(N)$ -action on spinors is the simple $N \times N$ matrix action $(Uz)_i = \sum_j U_{ij} z_j$. Defining the “homogeneous cosmological” sector as the $U(N)$ -invariant sector, satisfying $\langle z_i^\alpha | z_j^\alpha \rangle = \langle z_i^\beta | z_j^\beta \rangle$ and invariant under $z^\alpha, z^\beta \rightarrow Uz^\alpha, \bar{U}z^\beta$, imposes that all the α -spinors are equal to the β -spinors up to a global phase, $\bar{z}_i^{(\alpha)} = e^{i\phi} \bar{z}_i^{(\beta)}$. And we get a reduced phase space with two parameters, the total area λ and its conjugate angle ϕ encoding the curvature. Our ansatz for the dynamics of this “cosmological” sector is:

$$S_{inv}[\lambda, \phi] = -2 \int dt \left(\lambda \partial_t \phi - \lambda^2 \left(\gamma^0 - \gamma^+ e^{2i\phi} - \gamma^- e^{-2i\phi} \right) \right), \quad (3)$$

which corresponds to the quantum Hamiltonian defined above. In this classical case, the equations of motion can be solved exactly [5] with certain interesting analogies with (the effective dynamics of) loop quantum cosmology, showing that the dynamics of the $U(N)$ -invariant sector of the 2-vertex graph model can be interpreted as describing homogeneous and isotropic cosmology.

5. Conclusions

The $U(N)$ framework and the spinor representation introduced and studied in [1, 2, 3, 4, 5, 6, 7] represents a new and refreshing way to tackle several important issues in loop quantum gravity.

In this work we have discussed these new frameworks and we have reviewed a proposal for the dynamics of the homogeneous and isotropic sector of the model, both at the quantum (using the $U(N)$ framework) and the classical (using spinors) level. In this process, we have described the main features of the $U(N)$ framework, like the Fock space structure of the Hilbert space of intertwiners with N legs. We further used this $U(N)$ structure on the 2-vertex graph to define a symmetry reduction to the homogeneous and isotropic sector. We have then introduced a Hamiltonian consistent with this symmetry reduction, which can be solved exactly and shown to be analogous with the dynamics of loop quantum cosmology.

Acknowledgments

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